NOTATION

x, z, transverse and longitudinal coordinates; v_0, T_0, undisturbed velocity and temperature profiles; φ, θ, amplitudes of disturbances of stream function and temperature; λ, decrement; k, wave number.

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HYDRODYNAMIC STABILITY OF CONVECTIVE FLOW OF A NON-NEWTONIAN FLUID IN A VERTICAL LAYER

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Steady convective non-Newtonian fluid flow and its stability under small perturbations are investigated.

We wish to analyze the free thermal convection of a non-Newtonian fluid in an infinite-plane vertical channel. We use the rheological equation

$$\tau_{ij} = -\delta_{ij} p + \eta (1 + aI)^{n-1} \dot{e}_{ij}. \tag{1}$$

Transition to a Newtonian fluid takes place as $a \to 0$ or $n \to 1$. In the limit of large a the Ostwald–Deville model is obtained from (1). Unlike the power-law model, Eq. (1) gives a finite initial viscosity.

It has been shown [1] that Eq. (1) well describes the rheological properties of polymer solutions in a definite concentration interval. The authors of [1] discuss pseudoplastic media with $n - 1 = -m < 0$.

We now investigate plane convective motion homogeneous along the z axis. We place the coordinate axes so that the y axis is directed upward along the centerline of the channel and the x axis is perpendicular to the walls. The wall coordinates are $x = \pm h$. The walls are maintained at constant temperatures: $T(-h) = \Theta_0$; $T(h) = -\Theta_0$.

We adopt the following reference units: distance h; time $h^2\rho/\eta$; velocity $\rho g\beta\Theta_0 h^2/\eta$; temperature Θ_0; pressure $\rho g \beta \Theta_0 h$. The system of dimensionless free-convection equations in projections onto the x and y axes has the form

$$\frac{\partial v_x}{\partial t} + \mathrm{Gr}\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \left[H\Delta v_x + 2\frac{\partial H}{\partial x}\cdot\frac{\partial v_x}{\partial x} + \frac{\partial H}{\partial y}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right]; \tag{2}$$

$$\frac{\partial v_y}{\partial t} + \mathrm{Gr}\left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \left[H\Delta v_y + \frac{\partial H}{\partial x}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + 2\frac{\partial H}{\partial y}\cdot\frac{\partial v_y}{\partial y}\right] + T; \tag{3}$$

$$\frac{\partial T}{\partial t} + \mathrm{Gr}\,\vec{v}\nabla T = \mathrm{Pr}^{-1}\Delta T; \tag{4}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0; \tag{5}$$

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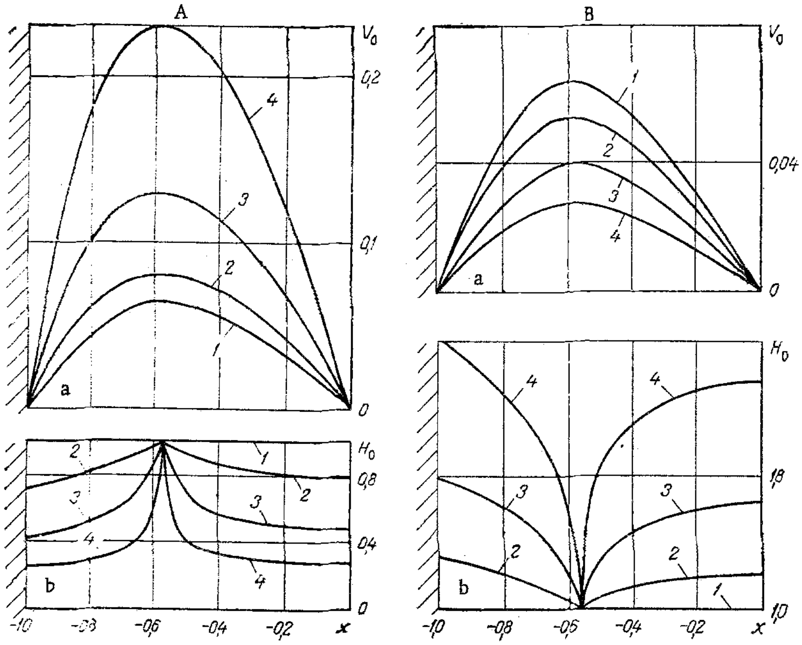


Fig. 1. Dimensionless velocity profiles in the left half of the channel cross section (a) for n = 0.8 (A) and n = 1.2 (B), and distribution of apparent viscosity (b) for the same values of n. 1) $\tilde{a}$ = 0; 2) 10; 3) 100; 4) 1000.

$$H = \left\{1 + \tilde{a}\left|\left[2\left(\frac{\partial v_x}{\partial x}\right)^2 + 2\left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2\right]^{\frac{1}{2}}\right|\right\}^{n-1}. \tag{6}$$

Here Gr = $\rho^2 g\beta\Theta_0 h^3/\eta^2$ and Pr = $\eta/(\chi\rho)$ are the Grashof and Prandtl numbers defined with respect to the initial viscosity η.

We find the velocity and temperature distributions for steady plane-parallel convective flow. We seek a solution of the system (2)-(6) with the structure

$$v_x = 0;\quad v_y = v_0(x);\quad T = T_0(x);\quad p = p_0(y). \tag{7}$$

Taking account of (7), we obtain from the system

$$\frac{dp_0}{dy} = \frac{d}{dx}\left[\left(1 + \tilde{a}\left|\frac{dv_0}{dx}\right|\right)^{n-1}\frac{dv_0}{dx}\right] + T_0 = A;\quad \frac{d^2T_0}{dx^2} = 0, \tag{8}$$

where A is variable-separation constant. The solution must satisfy the boundary conditions and flow closure condition

$$v_0 = 0;\quad T_0 = \pm 1 \quad \text{at} \quad x = \mp 1, \tag{9}$$

$$\int_{-1}^{1} v_0(x)\,dx = 0. \tag{10}$$

From expression (8) and the boundary conditions for T_0 we obtain the linear temperature profile

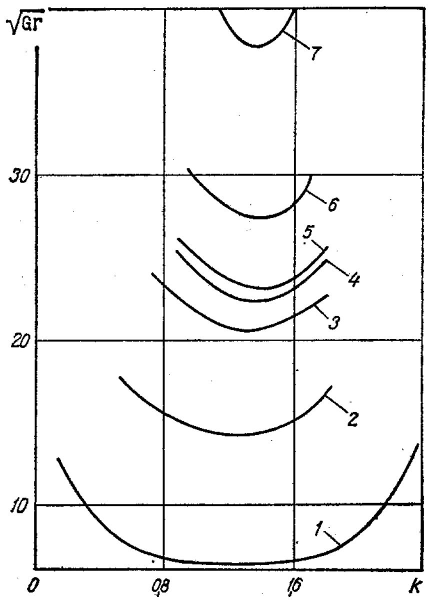
$$T_0 = -x. \tag{11}$$

The flow closure conditions imply that A = 0. Then for the velocity we obtain the equation

$$\frac{d}{dx}\left[\left(1 + \tilde{a}\left|\frac{dv_0}{dx}\right|\right)^{n-1}\frac{dv_0}{dx}\right] = x. \tag{12}$$

We solve Eq. (12) with the appropriate boundary conditions by an iterative procedure. We take as the initial approximation the solution obtained for this problem in the case of a power-law rheological model [2].

Figure 1A gives the dimensionless velocity profiles for n = 0.8 and various values of $\tilde{a}$ in one half of the channel cross section. For $\tilde{a}$ = 0 we have the familiar velocity profile for Newtonian flow [3]. Also given in this figure are the distributions of the "apparent viscosity" H_0 for the same values of the parameter $\tilde{a}$. As the latter is increased the average viscosity in the layer decreases. Consequently, the flow intensity increases.



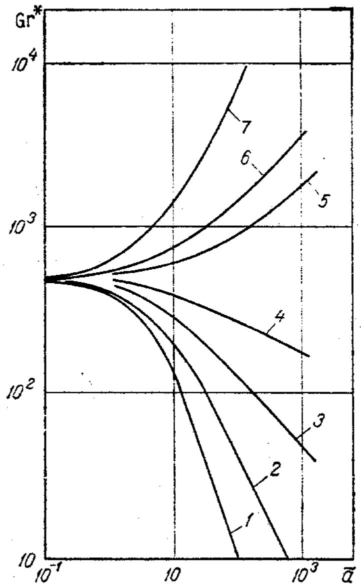


Fig. 2 Fig. 3

Fig. 2. Lower parts of stability neutral curves: $\mathrm{Gr}^{1/2}$ versus dimensionless wave number k for Pr = 1. 1) n = 0.7, $\tilde{a}$ = 100; 2) 0.7, 10; 3) 0.7, 1; 4) $\tilde{a}$ = 0, arbitrary n; 5) n = 1.2, $\tilde{a}$ = 1; 6) 1.2, 10; 7) 1.2, 100.

Fig. 3. Critical Grashof number Gr* versus dimensionless parameter $\tilde{a}$, both variables in logarithmic scale, for Pr = 1. 1) n = 0.6; 2) 0.7; 3) 0.8; 4) 0.9; 5) 1.1; 6) 1.2; 7) 1.6.

On the other hand, the nonuniformity of the viscosity distributed is heightened. With increasing $\tilde{a}$ a sharp maximum occurs at the point corresponding to maximum velocity; in the case of the power-law model ($\tilde{a} \to \infty$) a well-known singularity occurs at this point.

The analogous distributions for the dilatant case n = 1.2 are given in Fig. 1B. Here, by contrast, the average viscosity in the layer increases with increasing $\tilde{a}$, and the intensity of the motion subsides.

We also analyze the stability of the resulting stationary flow under small perturbations: $\vec{v} = \vec{v}_0 + \vec{v}'$; $p = p_0 + p'$; $T = T_0 + T'$. The primed variables denote small perturbations. The variables are substituted in this form into the system (2)-(6). Then the equations are linearized with respect to the small perturbations. We introduce the stream function $v'_x = -\partial\psi/\partial y$; $v'_y = \partial\psi/\partial x$. We consider normal perturbations of the form

$$\psi = \Phi(x)\exp(iky - \lambda t); \quad T' = \Theta(x)\exp(iky - \lambda t). \tag{13}$$

Here Φ and Θ are amplitudes, k is the wave number, and λ is the complex decay rate ($\lambda = \lambda_r + i\lambda_i$). Substituting (13) into the system of perturbation equations, we obtain for the amplitude equations

$$\lambda(\Phi'' - k^2\Phi) - \mathrm{Gr}\, v_0 ik(\Phi'' - k^2\Phi) + \mathrm{Gr}\, v_0'' ik\Phi + \left(A^{n-1} + \tilde{a}(n-1)A^{n-2}\left|\frac{dv_0}{dx}\right|\right)(\Phi^{\mathrm{IV}} + k^4\Phi) -$$
$$-2\left(A^{n-1} - \tilde{a}(n-1)A^{n-2}\left|\frac{dv_0}{dx}\right|\right)k^2\Phi'' + 2Rk^2\Phi' + 2(L+R)\Phi''' + \frac{d}{dx}(L+R)(k^2\Phi + \Phi'') + \Theta' = 0; \tag{14}$$

$$\mathrm{Pr}^{-1}(\Theta'' - k^2\Theta) + \lambda\Theta - ik\,\mathrm{Gr}(v_0\Theta - \Phi T_0') = 0. \tag{15}$$

Here we use the notation $A \equiv 1 + \tilde{a}\left|\frac{dv_0}{dx}\right|$; $R \equiv (n-1)\frac{d}{dx}(A^{n-2})\left|\frac{dv_0}{dx}\right|$; $L \equiv 2\frac{d}{dx}(A^{n-1})$. The prime indicates differentiation with respect to x. The following boundary conditions hold at rigid, perfectly heat-conducting channel walls:

$$\Phi = \Phi' = \Theta = 0 \quad \text{at} \quad x = \pm 1. \tag{16}$$

To determine the behavior of the perturbations we need to investigate the spectrum of eigenvalues of the boundary-value problem. The system (14)-(15) with the boundary conditions (16) is integrated by the Runge — Kutta — Merson method with orthogonalization in each integration step. The details of this method are given in [4].

An investigation of the analogous problem for a Newtonian fluid [3] has revealed two instability mechanisms: hydrodynamic and thermal. It is shown that the thermal instability arises and presents the greatest risk for large values of the Prandtl number; also, it has a wave nature. The objective of the present study is to isolate only the hydrodynamic instability branch for moderate values of the Prandtl number. Points of monotonic instability along the neutral curve correspond to values of $\lambda_r = \lambda_i = 0$.

Figure 2 shows the neutral curves for Pr = 1, n = 0.7 and n = 1.2, and various values of the parameter $\tilde{a}$. In the case of pseudoplastic flow (n = 0.7) the natural curve drops lower as $\tilde{a}$ is increased, i.e., destabilization takes place. In addition, the neutral curve changes shape, becoming flatter in the lower part. The critical wave number does not shift appreciably; its value remains around 1.3 or 1.4.

In the case of a dilatant fluid (n = 1.2), on the other hand, the critical Grashof number increases with the value of $\tilde{a}$.

The minimum Grashof number Gr* determines the stability threshold of steady plane-parallel flow, and its corresponding wave number determines the wavelength of the most dangerous perturbations. Curves of Gr* as a function of the parameter $\tilde{a}$ for various values of n are given in logarithmic scale in Fig. 3. As the curves indicate, increasing $\tilde{a}$ causes Gr* to decrease for n < 1 and to increase for n > 1. As $\tilde{a} \to 0$ the value of Gr* tends to 495 for any value of n, i.e., to the stability threshold for Newtonian fluid flow. With an increase in $\tilde{a}$ the Gr*($\tilde{a}$) curves depart from a log-linear dependence, i.e., the relationship between Gr* and $\tilde{a}$ goes over to a power law: $Gr* \sim \tilde{a}^m$. For $0.8 \leq n \leq 1$ the following asymptotic law holds: $Gr* \sim \tilde{a}^{2(n-1)}$.

As mentioned above, for large values of $\tilde{a}$ the transition eventually takes place to the power-law model. In this case the Grashof number Gr_{ef} defined with respect to the "effective" viscosity (see [6]) is the most indicative criterion. The relationship between Gr_{ef} and the Grashof number Gr defined with respect to the initial viscosity is as follows: $Gr_{ef} = (24\alpha)^2 Gr$, where $\alpha(n, \tilde{a})$ is the average dimensionless velocity over half the channel cross section. A recomputation of the critical values of Gr_{ef} from the data of Fig. 3 yields for large $\tilde{a}$ results that are consistent with those obtained earlier [6] for the power-law model in the effective-viscosity approximation.

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NOTATION

τ_{ij}, $\dot{e}_{ij}$, internal-stress and strain-rate tensors; η, consistency and initial viscosity; a, parameter of rheological equation; n, rheological power exponent; h, half-width of channel; Θ_0, wall temperature; ρ, fluid density; g, free-fall acceleration; β, temperature coefficient of volume expansion; χ, thermal diffusivity; $\vec{v}$, velocity vector; T, temperature; t, time; Gr, Grashof number; Pr, Prandtl number; ψ, stream function; k, wave number; λ, complex decay rate; Φ, amplitude of stream function; Θ, temperature perturbation amplitude function; $\tilde{a}$, dimensionless counterpart of rheological parameter a; P, pressure.

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